Math 1A – Calculus – Chapter 3 Test – Spring '08 Name________________________

Show your work for credit. Do not use a calculator.

1. On what interval is the function $f(x) = 7 - 3e^x + 5x$ increasing? Write your answer in interval notation.

2. Consider the parabola described by $y = 5x - x^2$
   a. Write an equation for the tangent line at (1,4).
   b. Where does the normal line at (1,4) cross the parabola again?
   c. Illustrate your solution to part (b) by sketch a graph for the parabola and normal line and two points of intersection.

3. If $u(2) = 7, v(2) = 3, u'(2) = -3,$ and $v'(2) = 2$, find the following numbers:
   a. $(uv)'(2)$
   b. $(u^2v)'(2)$

4. Consider the function $f(x) = x^2e^x$
   a. Write the second derivative $f''(x)$ as a product of e^x with some quadratic function.
   b. Over what interval(s) is $f(x)$ concave down?

5. A ball is given a push so that it has an initial velocity of 3 m/s down a certain inclined plane, and the distance it has rolled after t seconds is $s(t) = 8t + 3t^2$.
   a. Find the velocity after 2 seconds.
   b. How long does it take the velocity to reach 68m/s?

6. Find an equation for the tangent line to the curve $y = e^{2x}\sin(3x)$ at (0,0).

7. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it oscillates vertically. The equation of motion is $s(t) = 5\cos 2t + \sin 2t$, where s is measured in centimeters and t in seconds. (We take the positive direction to be downward.)
   a. Find the velocity at time t.
   b. Find the acceleration at time $t = \frac{\pi}{6}$

8. Find an equation for the line tangent to $y = e^{-x}\cos(2x)$ at (0,1).

9. Find $\frac{d^{19}}{dx^{19}}y$ for $y = xe^x$ by computing the first 8 derivatives and then following the pattern.

10. If $f(x) = \sin 3x\sqrt{1 - 2\sin^2 3x}$ find $f'(x)$

11. Find an equation for the line tangent to $3(x^2 + y^2)^2 = 100(x^2 - y^2)$ at (4,2)

12. Let $f(x) = \dfrac{\ln x}{x^5}$

a. Find $f'(x)$

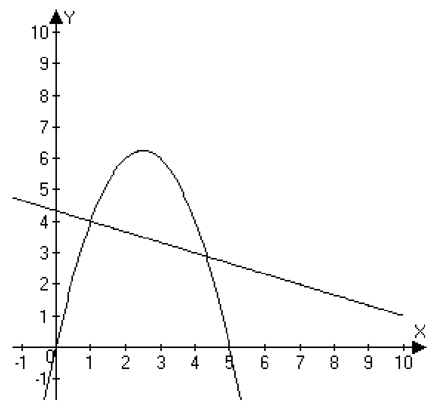
b. Find $f''(x)$

13. Use logarithmic differentiation to find $\dfrac{d}{dx} x^{10\cos x}$

14. Use a linear approximation to estimate $\dfrac{1}{99996}$.

Math 1A – Calculus – Chapter 3 Test Solutions – Spring '08

1. On what interval is the function $f(x) = 7 - 3e^x + 5x$ increasing? Write your answer in interval notation.
   SOLN: The function is increasing where the derivative is positive.
   $f'(x) = -3e^x + 5 > 0 \Leftrightarrow e^x < \frac{5}{3} \Leftrightarrow x < \ln\frac{5}{3}$ so the function is increasing on $\left(-\infty, \ln\left(\frac{5}{3}\right)\right)$.
2. Consider the parabola described by $y = 5x - x^2$
   a. Write an equation for the tangent line at (1,4).
   SOLN: The derivative function is $y' = 5 - 2x$ so the slope at $x = 1$ is $m = 3$ and the equation of the tangent line is $y = 4 + 3(x-1) = 3x + 1$
   b. Where does the normal line at (1,4) cross the parabola again?
   SOLN: The slope of the normal line is $m_\perp = -\frac{1}{3}$ so the equation is $y = 4 - \frac{1}{3}(x-1) = -\frac{1}{3}x + \frac{13}{3}$.
   At the points of intersection we have
   $5x - x^2 = -\frac{1}{3}x + \frac{13}{3} \Leftrightarrow 3x^2 - 16x + 13 = 0 \Leftrightarrow (x-1)(3x-13) = 0$
   So the x-coordinate of the other point is $x = \frac{13}{3}$. The y coordinate there is $y = \frac{26}{9}$.
   c. Illustrate your solution to part (b) by sketch a graph for the parabola and normal line and two points of intersection.
   SOLN: The x-intercepts of the parabola are at $x = 0$ and $x = 5$ and it opens downward from a vertex at $\left(\frac{5}{2}, \frac{25}{4}\right)$:



3. If $u(2) = 7, v(2) = 3, u'(2) = -3,$ and $v'(2) = 2$, find the following numbers:
   a. $(uv)'(2)$ SOLN:
   $u(2)v'(2) + u'(2)v(2) = 7(2) + (-3)(3) = 5$
   b. $(u^2v)'(2) = (u(2))^2 v'(2) + 2(u(2))u'(2)v(2) = 7^2 2 + 2(7)(-3)(3) = 98 - 126 = -28$
4. Consider the function $f(x) = x^2e^x$
   a. Write the second derivative $f''(x)$ as a product of e^x with some quadratic function.
   SOLN: $f'(x) = 2xe^x + x^2e^x = (x^2 + 2x)e^x$ so $f''(x) = (x^2 + 2x)e^x + (2x + 2)e^x = (x^2 + 4x + 2)e^x$
   b. Over what interval(s) is $f(x)$ concave down?
   SOLN: f is concave down if
   $$f''(x) < 0 \Leftrightarrow (x^2 + 4x + 2)e^x < 0 \Leftrightarrow x^2 + 4x + 2 < 0 \Leftrightarrow (x+2)^2 - 2 < 0$$
   $$\Leftrightarrow -2 - \sqrt{2} < x < -2 + \sqrt{2} \Leftrightarrow x \in \left(-2 - \sqrt{2}, -2 + \sqrt{2}\right)$$
5. A ball is given a push so that it has an initial velocity of 8 m/s down a certain inclined plane, and the distance it has rolled after t seconds is $s(t) = 8t + 3t^2$.
   a. Find the velocity after 2 seconds.
   SOLN: $s'(2) = 8 + 6t\big|_{t=2} = 20\,\text{m/s}$
   b. How long does it take the velocity to reach 68m/s?
   SOLN: $s'(2) = 8 + 6t = 68 \Leftrightarrow t = 10\,\text{sec}$

6. Find an equation for the tangent line to the curve $y = e^{2x}\sin(3x)$ at (0,0).

SOLN: $y' = (e^{2x})'\sin(3x) + e^{2x}(\sin(3x))' = 2e^{2x}\sin(3x) + 3e^{2x}\cos(3x)$

So the slope of the tangent line is $m = 0 + 3 = 3$ and the equation is $y = 3x$.

7. Find an equation for the line tangent to $y = e^{-x}\cos(2x)$ at (0,1).

SOLN: $y' = (e^{-x})'\cos(2x) + (e^{-x})(\cos(2x))' = -e^{-x}\cos(2x) - 2e^{-x}\sin(2x)$

At $x = 0$, the slope is $-1 + 0 = -1$. Thus the tangent line is $y = 1 - x$.

8. If $f(x) = \sin 3x\sqrt{1-2\sin^2 3x}$ find $f'(x)$.

SOLN:

$$
\begin{aligned}
f'(x) &= (\sin 3x)'\sqrt{1-2\sin^2 3x} + \sin 3x\left(\sqrt{1-2\sin^2 3x}\right)' \\
&= 3\cos 3x\sqrt{1-2\sin^2 3x} - \frac{\sin 3x(3(4)\sin 3x\cos 3x)}{2\sqrt{1-2\sin^2 3x}} \\
&= \frac{3\cos 3x(1-2\sin^2 3x) - 6\sin^2 3x\cos 3x}{\sqrt{1-2\sin^2 3x}} \\
&= \frac{3\cos 3x(1-4\sin^2 3x)}{\sqrt{1-2\sin^2 3x}}
\end{aligned}
$$

9. Find an equation for the line tangent to $3(x^2+y^2)^2 = 100(x^2-y^2)$ at (4,2)

SOLN: Equating derivatives of left and right sides, $6(x^2+y^2)(2x+2yy') = 100(2x-2yy')$ and then plugging in the coordinates,

$$6(16+4)(8+4y') = 100(8-4y') \Leftrightarrow 6(2+y') = 5(2-y') \Leftrightarrow 11y' = -2 \Leftrightarrow y' = -\frac{2}{11}$$

Thus the equation of the line is $y = 2 - \frac{2}{11}(x-4) = \frac{30}{11} - \frac{2}{11}x$

10. Let $f(x) = \frac{\ln x}{x^5}$

a. Find $f'(x)$: $f(x) = x^{-5}\ln x \Rightarrow f'(x) = (x^{-5})'\ln x + x^{-5}(\ln x)' = -5x^{-6}\ln x + x^{-6} = x^{-6}(1-5\ln x)$

b. Find $f''(x) = x^{-6}(1-5\ln x)' + (x^{-6})'(1-5\ln x) = -5x^{-7} - 6x^{-7}(1-5\ln x) = -x^{-7}(11-30\ln x)$

11. Use logarithmic differentiation to find $\frac{d}{dx}x^{10\cos x}$

SOLN:

$$
\begin{aligned}
&\ln y = 10\cos x\ln x \Rightarrow \frac{y'}{y} = \frac{10\cos x}{x} - 10\sin x\ln x \Rightarrow y' = x^{10\cos x}\left(\frac{10\cos x}{x} - 10\sin x\ln x\right) \\
&= (10\cos x)x^{10\cos x - 1} - 10\sin x(\ln x)x^{10\cos x}
\end{aligned}
$$

12. Use a linear approximation to estimate $\frac{1}{99996}$.

SOLN:

$$
\begin{aligned}
&f(x) = \frac{1}{x} \approx f(100000) + f'(100000)(x-100000) \\
&f(99996) \approx 10^{-5} - 10^{-10}(-4) = 1.00004\times 10^{-5}
\end{aligned}
$$

Note this is closer to $1.000040001600064002560102404096163846553862154486179447177 88\cdot 10^{-5}$